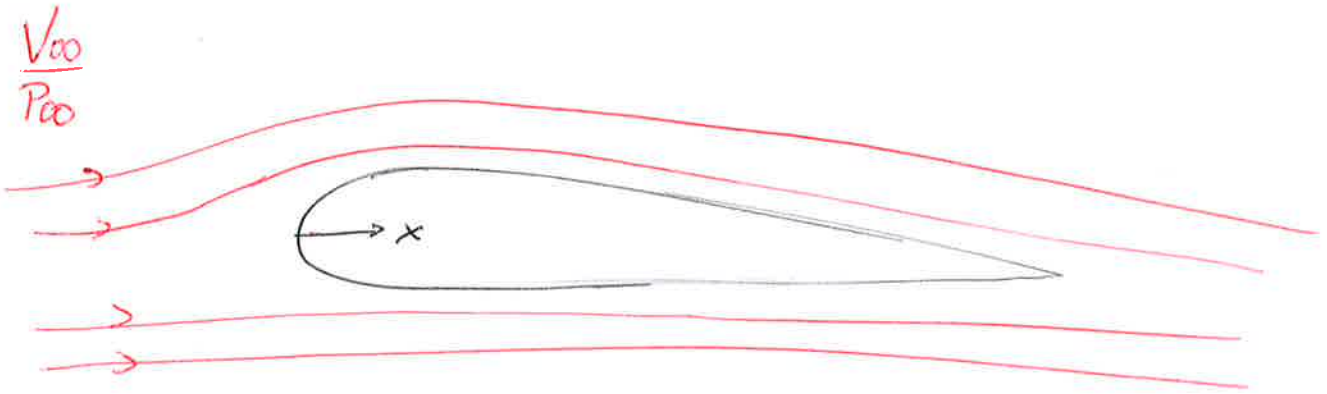
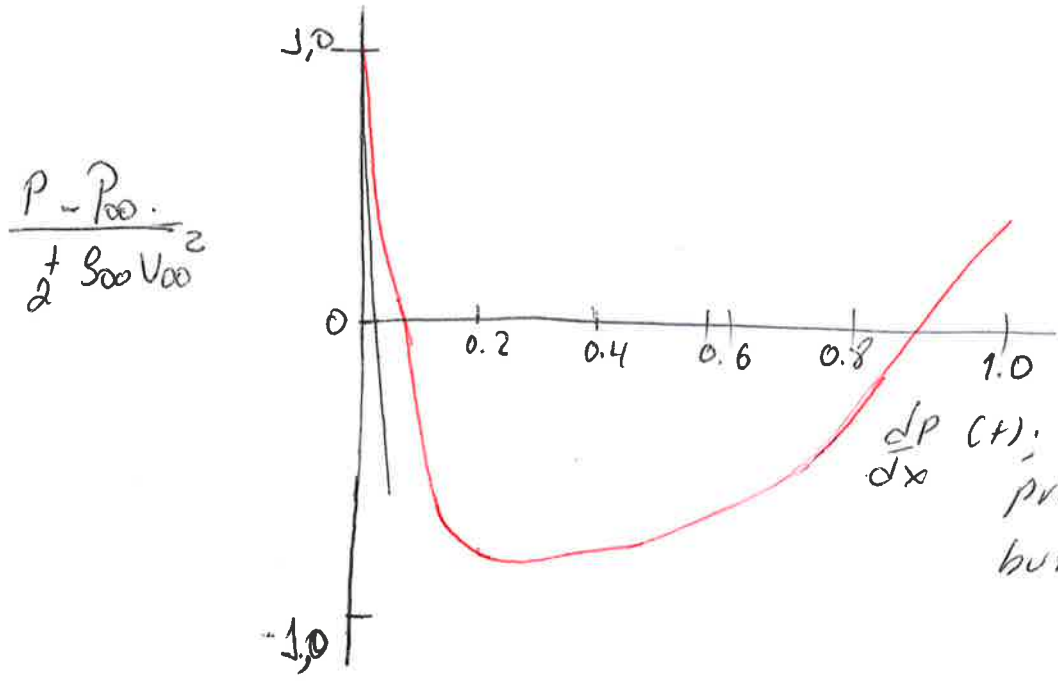


# Flow Separation



NACA LSCJ = 0417 airfoil  
 Angle of attack = 0°



$\frac{dp}{dx}$  (+). this is ~~an~~ adverse pressure gradient but is moderate.



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

Pressure drag on an airfoil is caused by the flow separation. For completely attached flow over an airfoil, the pressure acting on the rear <sup>surface</sup> ~~surface~~ gives rise to a force in the forward direction which completely counteract the pressure acting on the front surface producing a force rearward direction resulting zero pressure drag.

For a flow that is partially separated over the rear surface, the pressure on the rear surface pushing forward will be smaller than the fully attached case, and pressure drag is increased because the pressure acting on the front surface pushing backward is not counteracted.

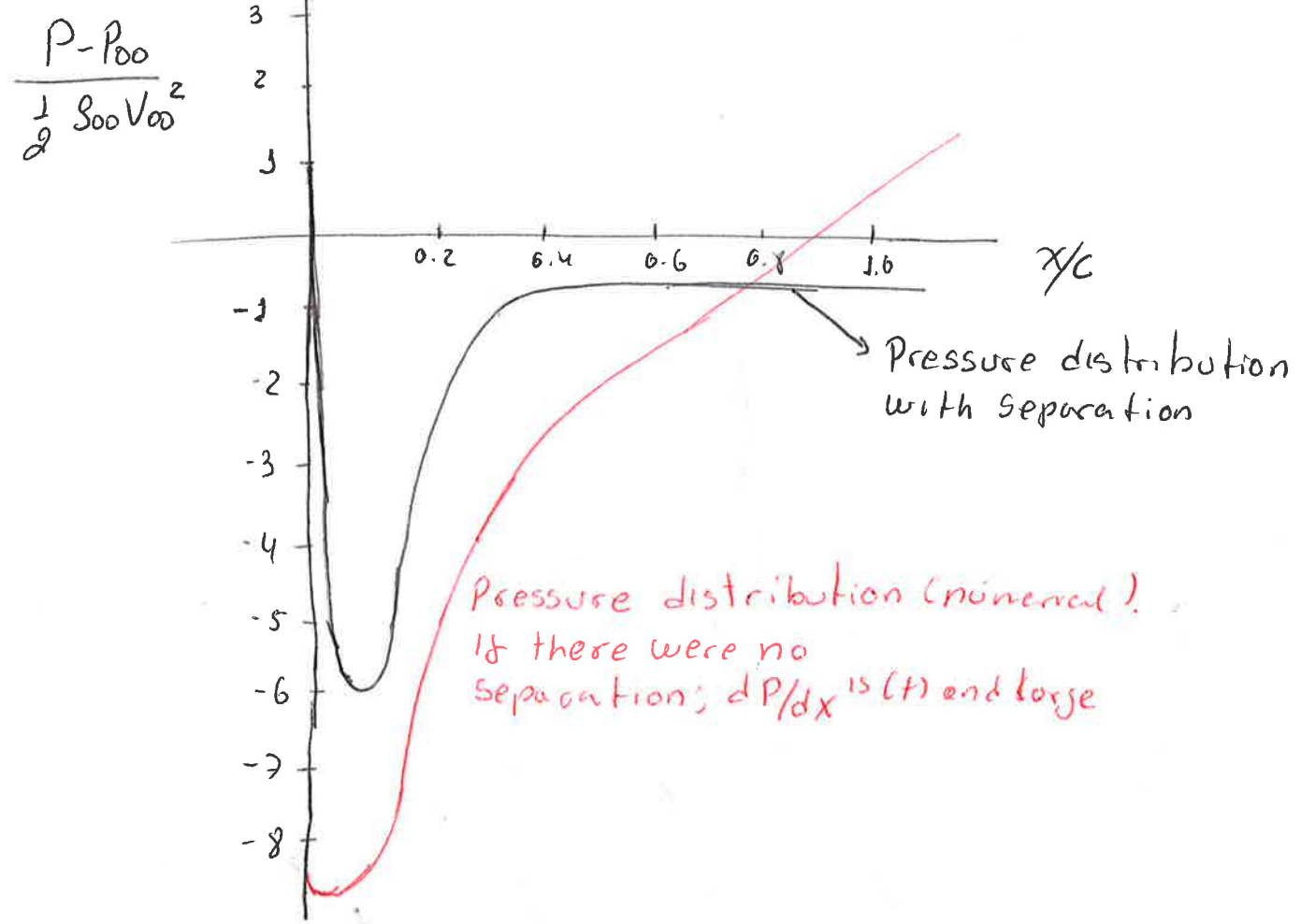
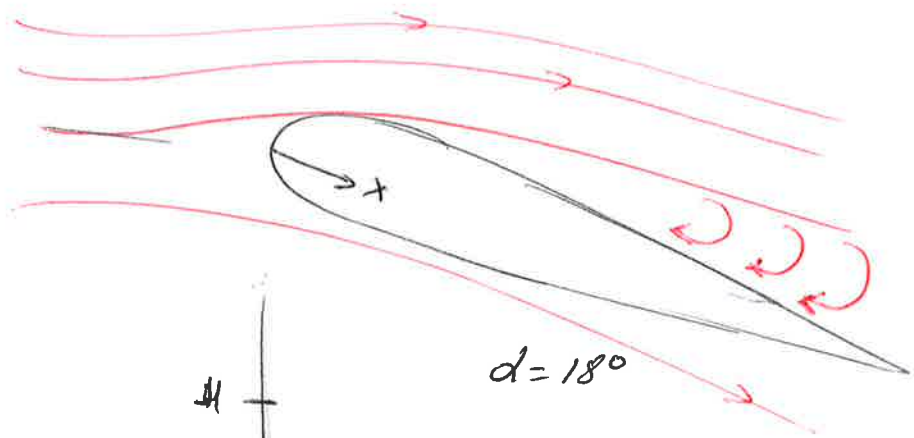
The region where the pressure increases, and reaching a value slightly above the free stream pressure at the trailing edge, is called the

**Cartagena99**

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70



The adverse pressure gradient would be severe: that is,  $dP/dx$  would be large. In such case,

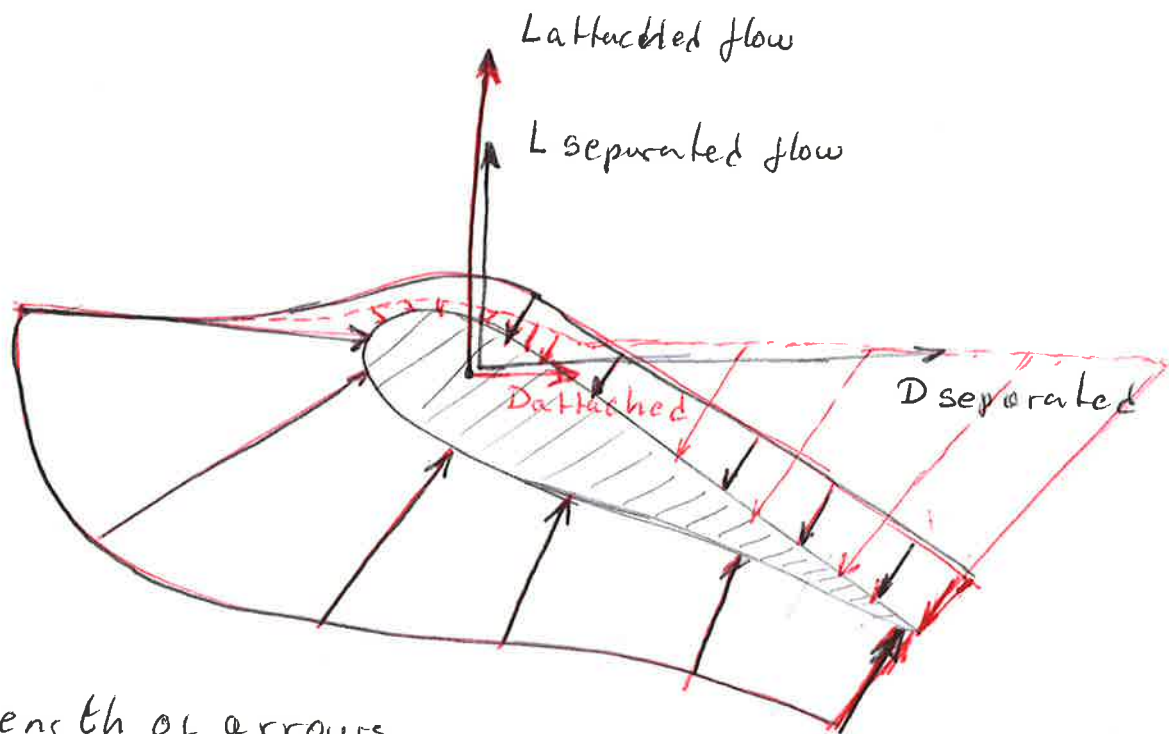


CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

In this real separated flow, the actual surface pressure distribution is given by solid curve, obtained by a computational fluid dynamic viscous flow calculation using the complete Navier-Stokes equations. In comparison to the numerical solution, the actual pressure distribution does not dip to as low a pressure minimum and the pressure near the trailing edge does not recover to a value above  $P_{\infty}$ .



Length of arrows denoting pressure is proportional to

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

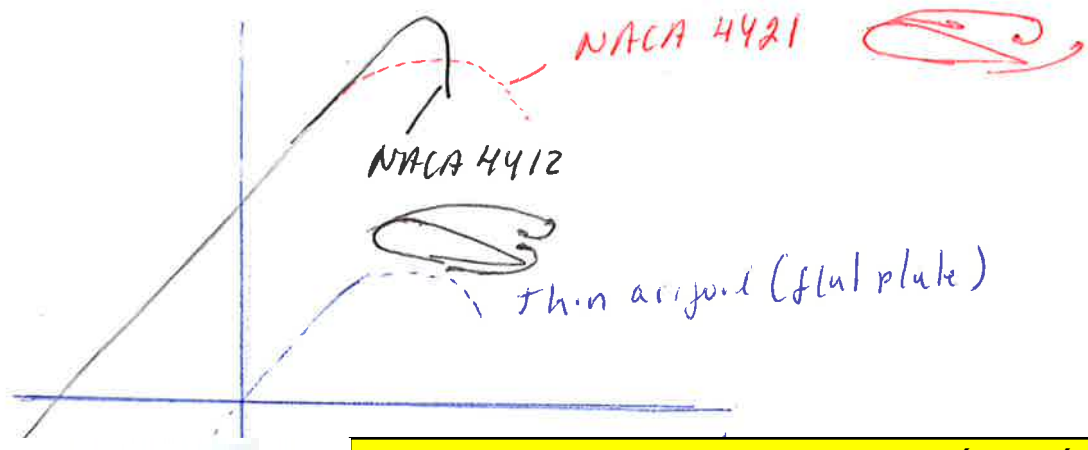
ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

Cartagena99

# Type of stalling phenomenon

**Leading edge stalling:** It is a characteristic relatively thin airfoils with thickness ratios between 10 or 16 percent of the chord length. Flow separation takes place rather suddenly and abruptly over entire top surface of the airfoil, with the origin of this separation occurring at the leading edge.

**Trailing edge stalling:** this behavior is characteristic of the thicker airfoils such as the NACA 4421, there is a progressive and gradual movement of separation from trailing edge toward the leading edge as  $\alpha$  is increased.



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70  
 ---  
 ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

# Aspects of airfoil aerodynamics

1. The lift-to-drag ratio  $L/D$ : An airfoil produces lift with minimum of drag;  $L/D$  is a measure of the aerodynamic efficiency of an airfoil.  $L/D$  ratio for a complete flight vehicle has an important impact on its flight performance.

2. The maximum lift coefficient  $C_{Lmax}$ : An effective airfoil produces a high value of  $C_{Lmax}$ . For a complete flight vehicle, the maximum lift coefficient  $C_{Lmax}$  determines the stalling speed of aircraft.

$$V_{stall} = \sqrt{\frac{2W}{\rho_{\infty} S C_{Lmax}}}$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70



# Incompressible Flow over Finite Wings

①

## Introduction:

From momentum considerations, that a vortex which is stationary with respect to a uniform flow experience a force of magnitude  $\rho V_{\infty} \Gamma$  in a direction perpendicular to  $V_{\infty}$ . The resulting Kutta-Joukowski theorem states that the force experienced by unit span of a right cylinder of any cross section whatever is  $\rho V_{\infty} \Gamma$  and it's directed perpendicular to  $V_{\infty}$ . It can state that a stationary line vortex normal to a moving stream is the equivalent of an infinite span wing (an airfoil) as far as the resultant force is concerned.

The airfoil-vortex analogy also forms the basis for calculating the properties of the finite wings, since the lift (and therefore, the

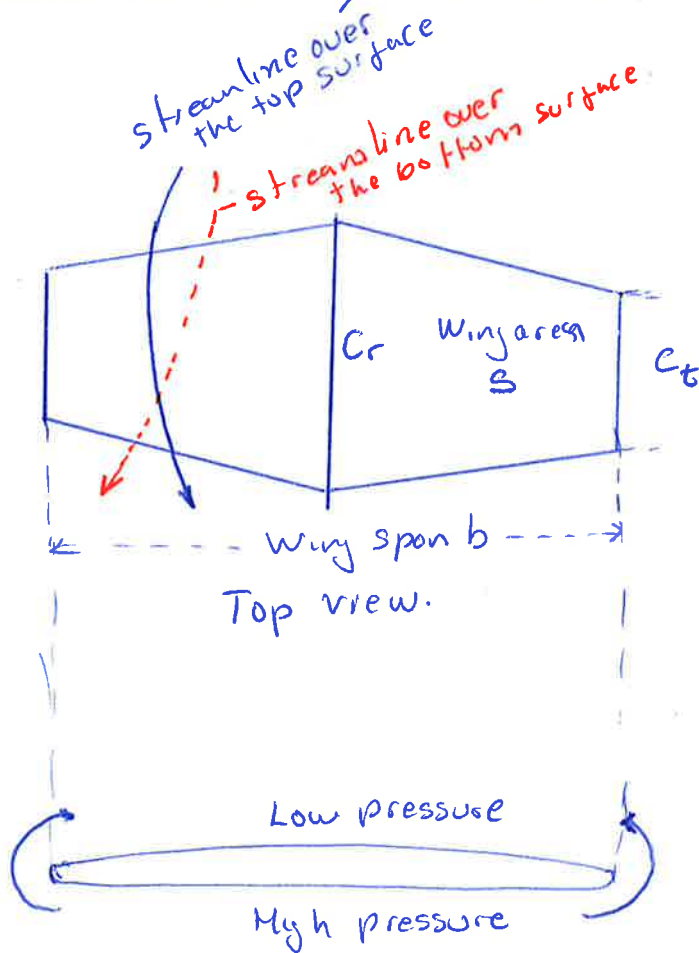
Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

An airfoil is simply a section of a wing, and at first thought, you might expect the wing to behave exactly the same as the airfoil. However the flow over an airfoil is two-dimensional. In contrast, a finite wing is a three-dimensional body and the flow over a finite wing is three-dimensional; that is, there is a component of flow in the spanwise direction.



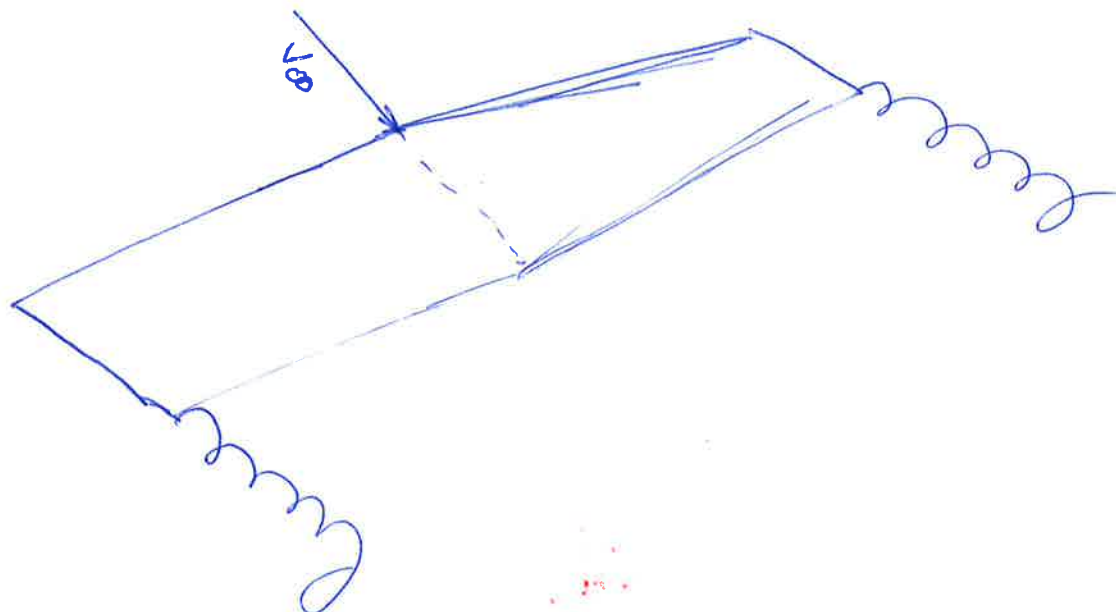
Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70





Wing-Tip vortices.

The tendency for the flow to "leak" around the wing tips has another important effect on the aerodynamics of the wing. This flow establishes a circulatory motion that trails downstream of the wing; that is, a trailing vortex is created at each wing tip. Tip vortices are essentially weak "tornadoes" that trail downstream of the finite wing. These wing-tip vortices downstream of the wing induce a small downward component of air velocity in the neighborhood of the wing itself.

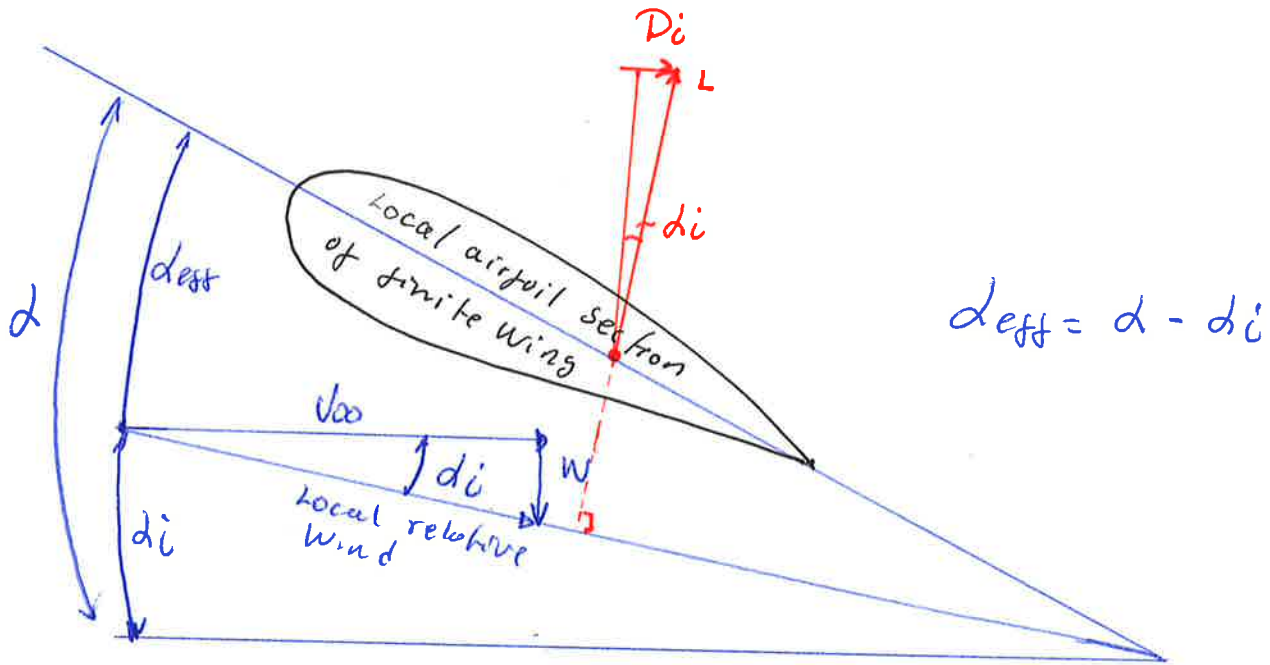
Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

This downward direction is called downwash. Denoted by  $(W)$ . In turn, the downwash combines with the freestream velocity  $V_{\infty}$  produce a local relative wind which is coated downward in the vicinity of each airfoil section wing



1. The angle of attack actually seen by the local airfoil section is the angle between the chord line and the local relative wind.  $\alpha_{eff}$ , effective angle of attack.

$$\alpha_{eff} = \alpha - di$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

of  $V_{\infty}$ .

there is a Drag created by the presence of downwash. This drag is defined as induced drag. (3)

1. The three-dimensional flow induced by the wing-tip vortices, simply alters the pressure distribution on the finite wing, a net pressure imbalance exists in the direction of  $V_{\infty}$ . In this sense, induced drag is a type of pressure drag.

2. The wing-tip vortices contain a large amount of translational and rotational kinetic energy. Since the energy of the vortices serves no useful purpose, this power is essentially lost. So the extra power provided by the engine that goes into vortices is the extra power required from the engine to overcome the induced drag.

The total drag on a subsonic finite wing in real life is the sum of the induced  $D_i$ , the skin friction  $D_f$  and the pressure drag  $D_p$ , the latter two contributions come from the

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

$$C_d = \frac{D_f + D_p}{\rho \infty S}$$

Profile drag coefficient for a finite wing. (moderate  $\alpha$ )

and the induced drag coefficient as.

$$C_{Di} = \frac{D_i}{\rho \infty S}$$

Total Drag coefficient for the finite wing is given by

$$C_D = C_d + C_{Di}$$

Usually  $C_d$  is obtained from airfoil data, such as an experimental data for profile drag coefficient. the value of  $C_{Di}$  can be obtained from finite wing theory.

The logo for Cartagena99, featuring the text 'Cartagena99' in a stylized, green, serif font with a blue and orange gradient background.

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

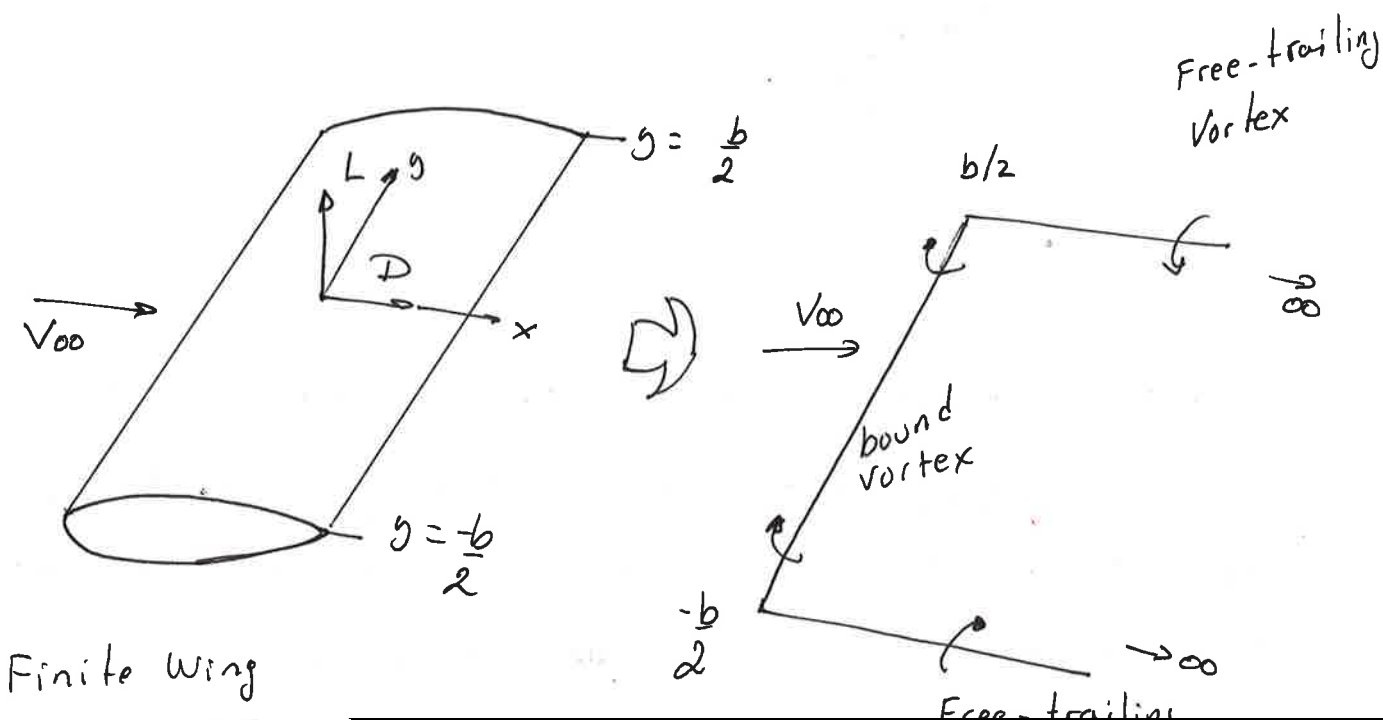
ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

# Prandtl's classical Lifting-Line theory

Ludwig Prandtl's was the first on developing a theory to predict the aerodynamic properties of finite wing.

He stated that a vortex filament of strength  $\Gamma$  that is somehow bound to a fixed location in a flow. (bound vortex) will experience a force.

$L = \rho \infty V \infty \Gamma$ . This bound vortex is in contrast to a free vortex, which moves with the same fluid elements throughout a flow.

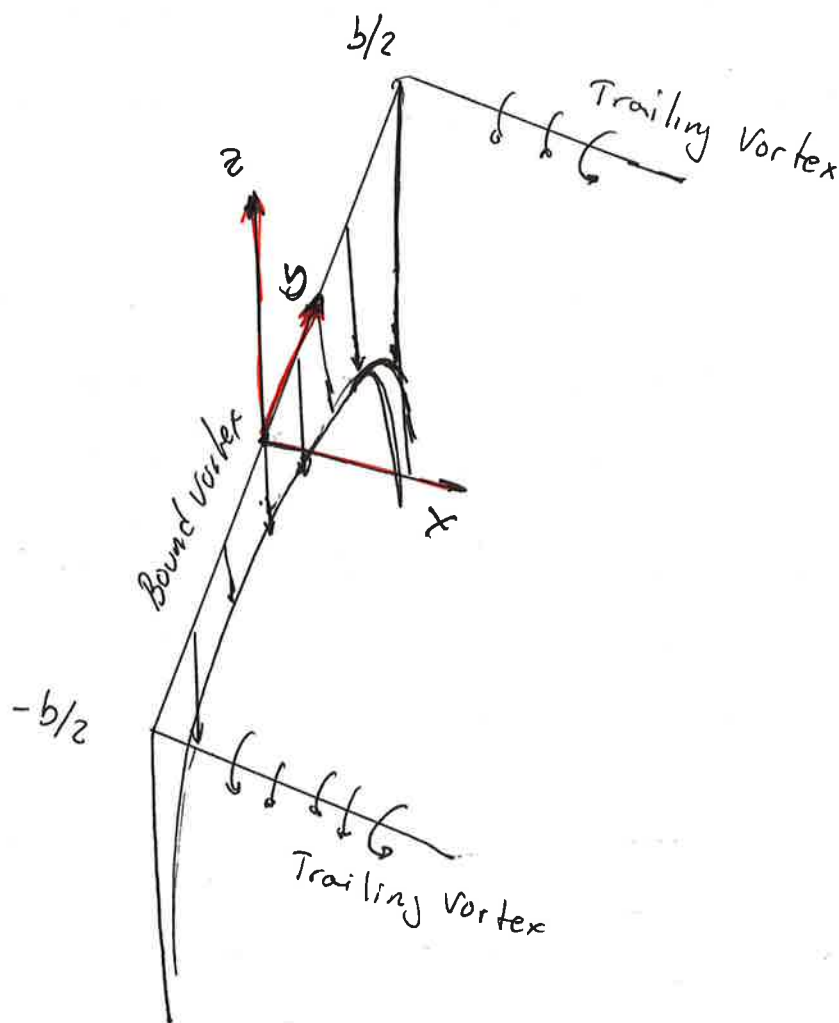


CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

Consider the downwash  $w$  induced along the bound vortex from  $-\frac{b}{2}$  to  $\frac{b}{2}$  by the horseshoe vortex.



The two trailing vortices both contribute to induced velocity along the bound vortex, and these contributions are in the downward direction. The direction is negative ( $w$ ) ( $z$  direction). The

Cartagena99

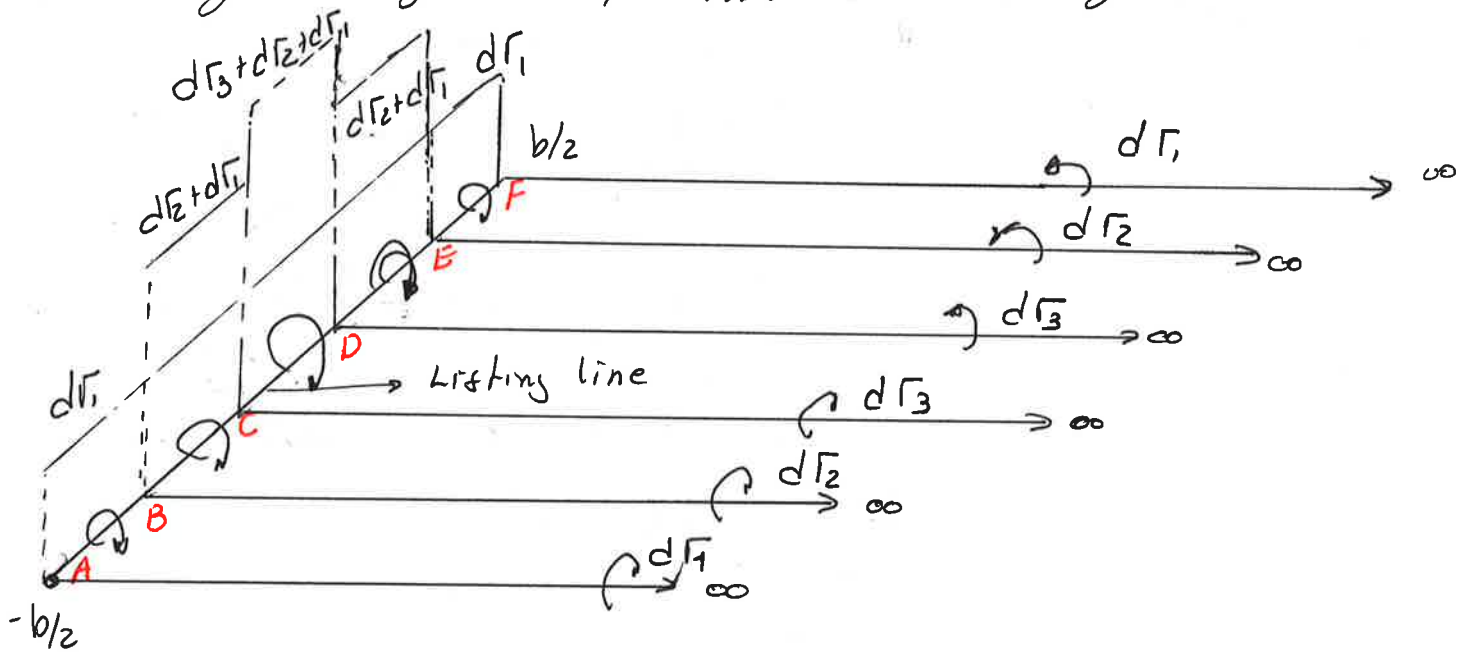
CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70



at  $y = \frac{b}{2}$  and  $y = -\frac{b}{2}$   $w(y) \rightarrow \infty$ , that is not realistic solution. Instead of representing the wing by a single horseshoe vortex. This can be represented by a large number of horseshoe vortices, each with different length of bound vortex, but with all the bound vortices coincident along a single line, called the lifting line.



Let us extrapolate to the case where an infinite number of horseshoe vortices are superimposed along the lifting line, each with a vanishingly small strength  $d\Gamma$ .

Cartagena99

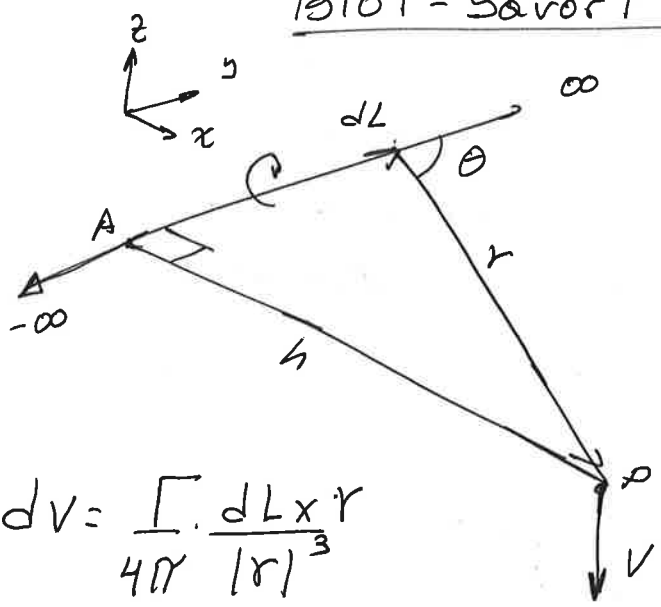
CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

# Biot-Savart Law

9



$$dV = \frac{\Gamma \cdot dL \times r}{4\pi r^3}$$

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\sin \theta}{r^2} dL$$

$$r = \frac{h}{\sin \theta}$$

$$L = \frac{h}{\tan \theta}$$

$$dL = -\frac{h}{\sin^2 \theta} d\theta$$

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\sin \theta}{r^2} dL$$

$$V = -\frac{\Gamma}{4\pi h} \int \sin \theta d\theta$$

$\left[ V = \frac{\Gamma}{2\pi h} \right]$  For infinite filament  
 For semi-infinite is half.

$\left[ V = \frac{\Gamma}{4\pi h} \right]$  Semi-Infinite

Applying for our discussion.  $V = w(y)$

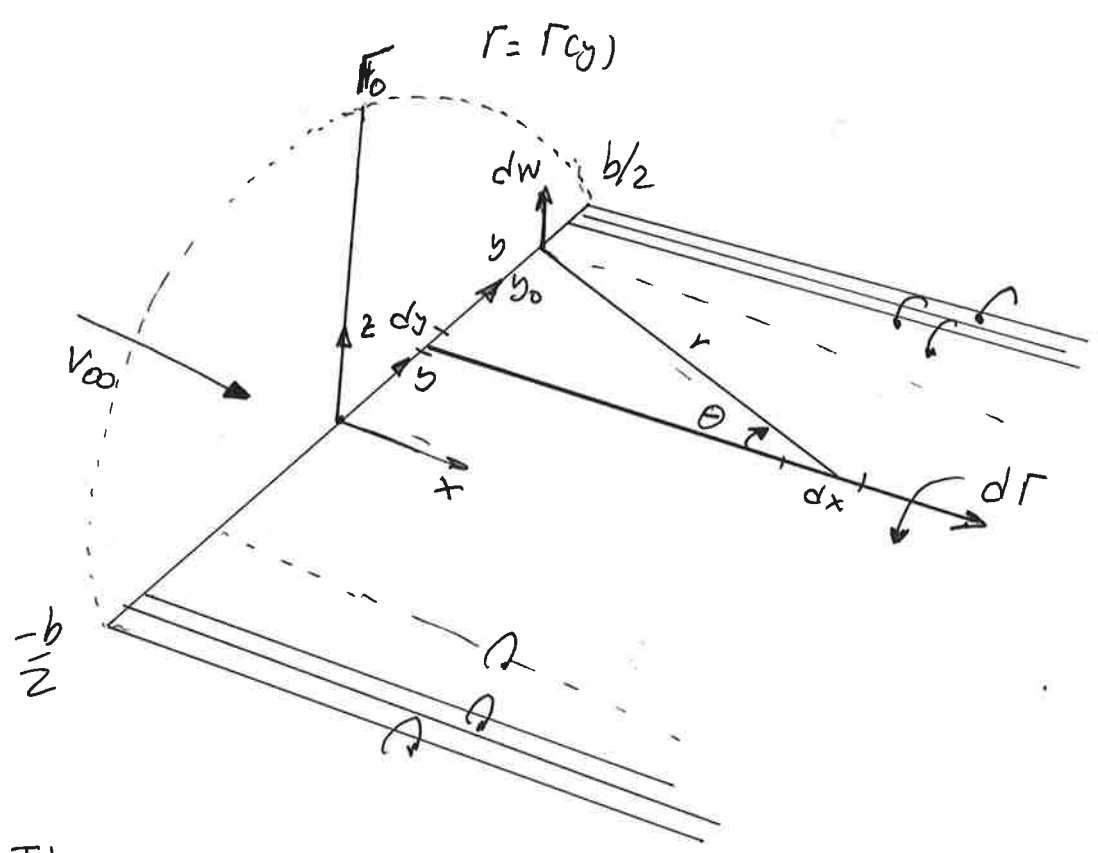
$$w(y) = -\frac{\Gamma}{4\pi \left(\frac{b}{2} + y\right)} - \frac{\Gamma}{4\pi \left(\frac{b}{2} - y\right)}$$



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70



The value of the circulation at origin is  $\Gamma_0$ . The finite number of trailing vortices have become a continuous vortex sheet trailing downstream of the ~~the~~ <sup>lifting</sup> line. This vortex sheet is parallel to the direction of  $V_{\infty}$ . The total strength of the sheet integrated across the span of the wing is zero, because it consists of pairs of trailing vortices of equal strength but in opposite direction.

From the figure we note that  $\Gamma(y)$  function of  $y$ . then  $d\Gamma = (d\Gamma/dy)dy$ . we obtain  $dw$  as



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

The total velocity  $w$  induced at  $y_0$  by the entire trailing vortex sheet is the summation of the above equation.

$$w(y_0) = -\frac{\Gamma}{4\pi} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy) \cdot dy}{(y_0 - y)}$$

We want to calculate  $\Gamma(y)$  for a given finite wing, along with its corresponding total lift and induced drag.

The above equation is important in that it gives the value of the downwash at  $y_0$  due to all the trailing vortices.

Suppose that local airfoil section along the wing span is at  $y_0$ .

As  $V_{\infty}$  is bigger than  $w$ ,  $\alpha_i$  is small.

$$\alpha_i(y_0) = \tan^{-1} \left( \frac{-w(y_0)}{V_{\infty}} \right)$$

For small angle we obtain that

$$\alpha_i(y_0) = \frac{-w(y_0)}{V_{\infty}}$$

**Cartagena99**

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

then  $d_i$  is obtained as:

$$d_i(y_0) = \frac{1}{4\pi V_{\infty}} \int_{-b/2}^{b/2} \frac{d\Gamma/dy}{y_0 - y} dy$$

That is, an expression for induced angle of attack in terms of circulation  $(\Gamma(y))$  along the wing.

Consider again the effective angle of attack  $d_{eff}$ , this angle of attack is seen by the local airfoil section. As  $d_i$  varies across the span, then  $d_{eff}$  is also variable, thus  $d_{eff} = d_{eff}(y_0)$ .

The lift coefficient for the airfoil section located at  $y = y_0$  is

$$C_L = d_0 [d_{eff}(y_0) - d_{L=0}] = 2\pi [d_{eff}(y_0) - d_{L=0}]$$

The local section lift slope  $d_0$  has been replaced by the thin airfoil theoretical value of  $2\pi (c_m - 2)$



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

If there is no aerodynamic twist  $d_{L=0}$  is constant across the span.  $d_{L=0}$  is a known property of the local airfoil section. From definition the local airfoil section lift at  $y_0$  is.

$$L = \rho_{\infty} V_{\infty} \Gamma(y_0) = \frac{1}{2} \rho_{\infty} V_{\infty}^2 C_L(y_0) \cdot C_L$$

$$C_L = \frac{2 \Gamma(y_0)}{V_{\infty} C(y_0)} \quad \text{and} \quad C_L = 2\pi [d_{eff}(y_0) - d_{L=0}]$$

From these equations we obtain  $d_{eff}$

$$\frac{2 \Gamma(y_0)}{V_{\infty} C(y_0)} = 2\pi [d_{eff}(y_0) - d_{L=0}]$$

$$\left[ d_{eff} = \frac{\Gamma(y_0)}{\pi V_{\infty} C(y_0)} + d_{L=0} \right]$$

and  $d_{eff} = d - d_i$

$$d(y_0) = \frac{\Gamma(y_0)}{\pi V_{\infty} C(y_0)} + d_{L=0}(y_0) + \frac{1}{4\pi V_{\infty}} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy) dy}{y_0 - y}$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70



This states that the geometric angle of attack <sup>⑧</sup> is equal to the sum of the effective angle plus the induced angle of attack. def out  $d_i$  are expressed by  $\Gamma$  and  $d\Gamma/dy$  respectively. Hence, the only unknown is  $\Gamma$ ; all the other quantities  $d$ ,  $C$ ,  $V_{\infty}$  and  $d_{L=0}$  are known for a finite wing of given design at a given geometric angle of attack in a freestream with given velocity. So a solution of the above equation yields  $\Gamma = F(y_0)$  and  $-\frac{b}{2} < y_0 < \frac{b}{2}$

The solution  $\Gamma = \Gamma(y_0)$  obtained gives the three main aerodynamic characteristics of finite wing.

1.) Lift distribution is obtained from Kutta-Joukowski

$$L(y_0) = \rho_{\infty} V_{\infty} \Gamma(y_0)$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

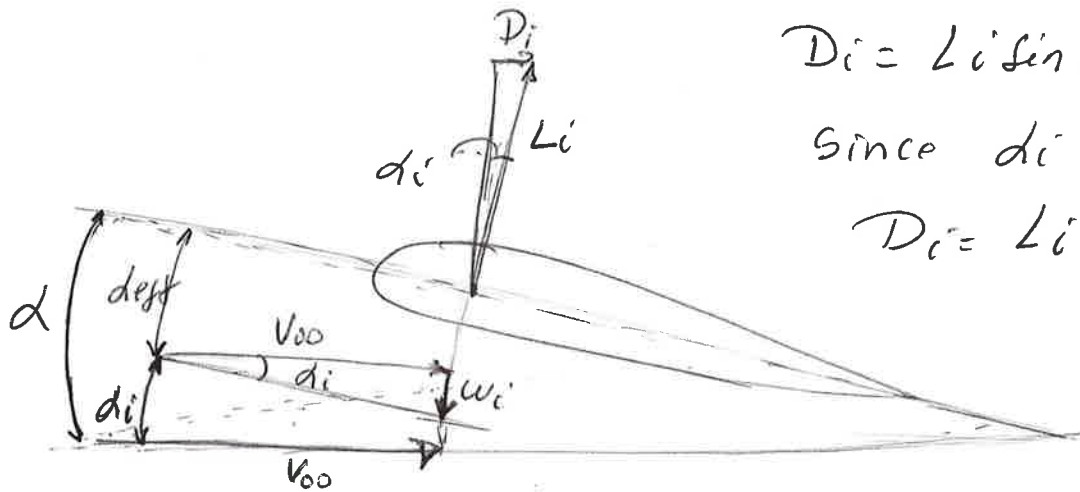
ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

$-\frac{b}{2}$

The lift coefficient follows immediately from the equation

$$C_L = \frac{L}{\rho \infty S} = \frac{2}{\rho \infty S} \int_{-b/2}^{b/2} \Gamma(y) dy$$

3). The induced drag is obtained by inspection of the figure.



$$D_i = L_i \sin \alpha_i$$

Since  $\alpha_i$  is small

$$D_i = L_i \alpha_i$$

$$D_i = \int_{-b/2}^{b/2} L(y) \cdot \alpha_i(y) dy$$

$$D_i = \rho \infty V_{\infty} \int_{-b/2}^{b/2} \Gamma(y) \alpha_i(y) dy$$

$$C_{D_i} = \frac{D_i}{\rho \infty S V_{\infty}^2} = \frac{2}{\rho \infty S V_{\infty}^2} \int_{-b/2}^{b/2} \Gamma(y) \alpha_i(y) dy$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

# Elliptical Lift Distribution

A very important special case is the elliptical circulation distribution, this distribution represents the wing of minimum induced drag. Fortunately, the properties of wings of arbitrary planforms that do not differ radically from the usual shapes are close to those of the elliptical wing.

Consider a circulation distribution given by

$$\Gamma = \Gamma_0 \cdot \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

$\Gamma_0$ : circulation at the origin

The circulation varies elliptically with distance  $y$  along the span; for that, it is designated as an elliptical circulation distribution.



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

$\Gamma(b/2) = \Gamma(-b/2) = 0$ . So, the circulation, hence, Lift goes from zero at wing tip.

First, let's calculate the downwash.

$$w(y_0) = \frac{\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{dy}{(1 - 4y^2/b^2)^{1/2} (y_0 - y)}$$

$$\frac{d\Gamma}{dy} = -\frac{4\Gamma_0}{b^2} \frac{y}{(1 - \frac{4y^2}{b^2})^{1/2}}$$

We substitute  $y = \frac{b}{2} \cos \theta$  and  $dy = -\frac{b}{2} \sin \theta d\theta$

$$w(\theta_0) = -\frac{\Gamma_0}{2\pi b} \int_{\pi}^0 \frac{\cos \theta}{\cos \theta - \cos \theta_0} d\theta$$

$$w(\theta_0) = -\frac{\Gamma_0}{2\pi b} \int_0^{\pi} \frac{\cos \theta}{\cos \theta_0 - \cos \theta} d\theta$$

$$\left[ w(\theta_0) = \frac{-\Gamma_0}{2b} \right]$$

This solution states that the downwash is constant across the span.

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

Cartagena99

And we calculate  $d_i$  based on  $d_0$  is small (10)

$$d_i = -\frac{W}{V_{\infty}} = -\left(\frac{-\Gamma_0}{2bV_{\infty}}\right) \frac{1}{2bV_{\infty}} = \frac{\Gamma_0}{2bV_{\infty}}$$

Indicates that  $d_i$  at any point along the lifting line is constant if the  $\Gamma$  distribution is elliptical.

$$L(y) = \rho_{\infty} V_{\infty} \Gamma_0 \int_{-b/2}^{b/2} (1 - 4\frac{y^2}{b^2})^{1/2} dy$$

$$L(y) = \rho_{\infty} V_{\infty} \Gamma_0 \cdot \frac{b}{2} \int_0^{\pi} \sin^2 \theta d\theta$$

$$L(y) = \rho_{\infty} V_{\infty} \Gamma_0 \frac{b}{4} \pi$$

$$\Gamma_0 = \frac{4L}{\rho_{\infty} V_{\infty} b \pi} \quad \text{and} \quad L = \frac{1}{2} \rho_{\infty} V_{\infty}^2 S \cdot C_L$$

$$\Gamma_0 = \frac{4 \cdot \frac{1}{2} \rho_{\infty} V_{\infty}^2 S \cdot C_L}{\rho_{\infty} V_{\infty} b \cdot \pi} = \frac{2 \cdot V_{\infty} S C_L}{b \pi}$$

**Cartagena99**

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

One important geometric property of a finite wing is the aspect ratio,

$$AR = \frac{b^2}{S}$$

Hence, the above equation (induced angle of attack) is given by

$$d_i = \frac{C_L}{\pi AR}$$

The induced drag coefficient is obtained from

$$C_{Di} = \frac{2d_i}{V_{\infty} S} \int_{-b/2}^{b/2} \Gamma(y) dy = \frac{2d_i \rho_{\infty} b}{V_{\infty} S} \int_0^{\pi} \sin^2 \theta d\theta$$

$$C_{Di} = \frac{\pi}{2} \frac{d_i \rho_{\infty} b}{V_{\infty} S}$$

$$C_{Di} = \frac{\pi b}{2 V_{\infty} S} \times \left( \frac{C_L}{\pi AR} \right) \times \frac{2 V_{\infty}^2 S C_L \rho_{\infty}}{\rho_{\infty} V_{\infty} b \pi}$$

$$\left[ C_{Di} = \frac{C_L^2}{\pi AR} \right]$$

The Drag coefficient (induced) is directly proportional to the square of the lift

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

a substantial part of the total drag when  $C_L$



Ex.

An untwisted wing with an elliptical planform and elliptical lift distribution has an aspect ratio of 6 and a span of 12m. The wing loading is 900 N/m<sup>2</sup> when flying at a speed of 150 km/h. (41.67 m/s) at sea level. We shall compute the induced drag for this wing.

The projected area and the total lift of the wing are, respectively.

$$S = \frac{b^2}{AR} = 24 \text{ m}^2 \quad L = 21600 \text{ N} = 900 \times 24 \frac{\text{N}}{\text{m}^2} \cdot \text{m}^2$$

the dynamic pressure is

$$q_{\infty} = \frac{1}{2} (1.226) \frac{\text{kg}}{\text{m}^3} (41.67)^2 \frac{\text{m}^2}{\text{s}^2} =$$

$$q_{\infty} = 1064 \text{ N/m}^2$$

$$C_l = C_d = \frac{L}{q_{\infty} S} = 0.846$$



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70  
 ---  
 ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

the induced drag is

$$D_i = L \frac{C_{Di}}{C_L} = 970 \text{ N}$$

which is 4.49% of the total lift.

The additional power that is required to compensate for the induced drag of this finite wing

$$P = D_i V_{\infty} = 970 \text{ N} \cdot \frac{41.67 \text{ m}}{5} = 54.2 \text{ hp}$$

The induced angle of attack and the constant downwash.

$$\alpha_i = -\frac{C_{Di}}{C_L} = -0.045 \text{ rad. } (-2.58^\circ)$$

$$w = -\frac{d\alpha_i}{dt} \cdot V_{\infty} = 0.045 \times 41.67 = 1.88 \text{ m/s}$$

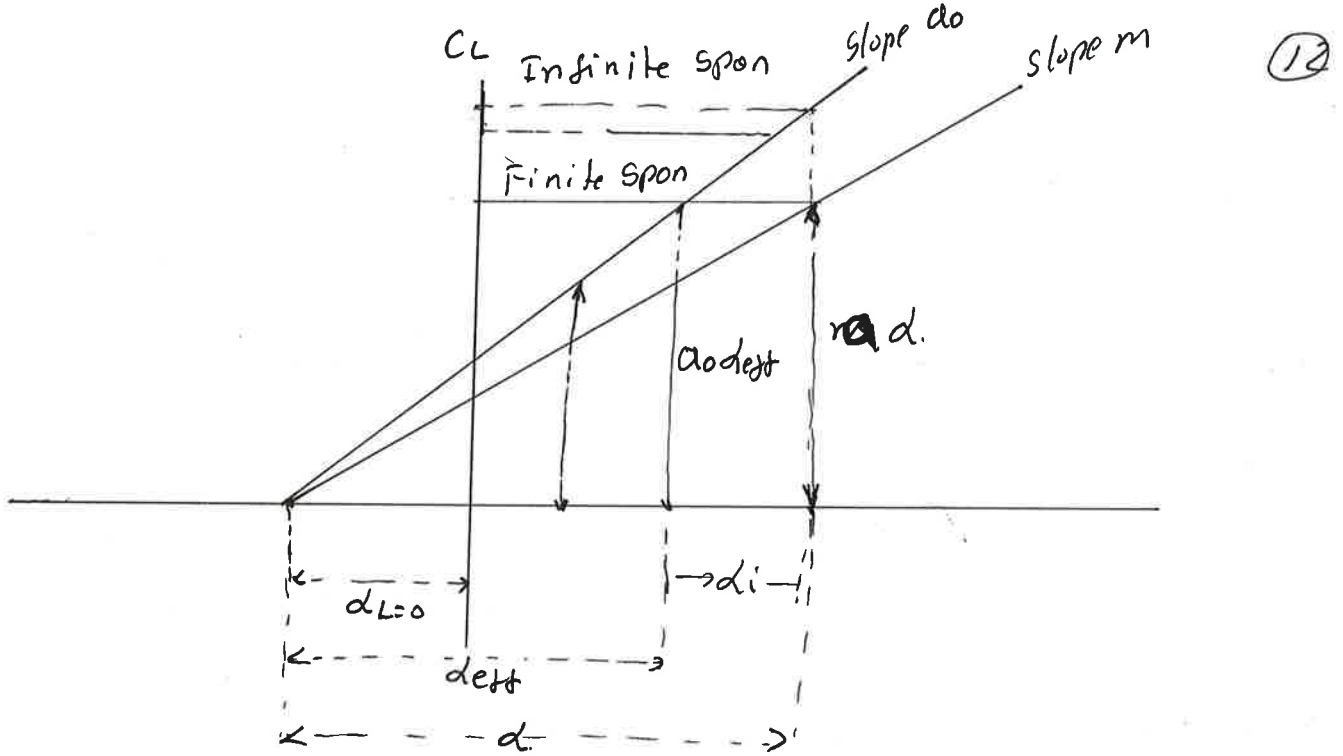
The effective angle of attack and the absolute angle of attack are constant.

**Cartagena99**

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70



$$d = d_{eff} + d_i$$

$$\frac{C_L}{\alpha} = \frac{C_L}{\alpha_0} + d_i$$

$$|\alpha| = \frac{\alpha_0}{1 + \frac{d_i \cdot \alpha_0}{C_L}} = \frac{\alpha_0}{1 + \frac{d_i}{d_{eff}}}$$

$$\left[ \alpha = \frac{\alpha_0}{1 + \frac{0.045}{0.135}} = 0.75 \alpha_0 \right]$$

This shows that the finite wing, in this example, generates only 75% of the lift that would be generated by the same wing if the effect of induced downwash were ignored.

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

Cartagena99

The integral is  $\int_0^{\pi} \sin(n\theta) \cdot \sin\theta d\theta = \begin{cases} \pi/2 & \text{for } n=1 \\ 0 & \text{for } n \neq 1 \end{cases}$

$$C_L = A_1 \cdot \pi \cdot \frac{b^2}{5} = A_1 \cdot \pi \cdot AR \quad (\text{only depends on the leading coefficient of the Fourier series expansion})$$

Then we must solve for all the  $A_n$ 's simultaneously in order to obtain  $A_1$ .

The induced drag coefficient is obtained from

$$C_{Di} = \frac{2}{V_{\infty}^2} \int_{-b/2}^{b/2} \Gamma(y) \cdot d_i(y) dy$$

$$= \frac{2b^2}{5} \int_0^{\pi} \left( \sum_1^N A_n \cdot \sin(n\theta) \right) \cdot d_i(\theta) \cdot \sin\theta d\theta$$

where  $d_i(\theta)$  is obtained by

$$d_i(y_0) = \frac{1}{4\pi V_{\infty}} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy) \cdot dy}{y - y_0}$$

$$d_i(\theta_0) = \frac{1}{\pi} \cdot \sum_1^N n \cdot A_n \cdot \int_0^{\pi} \frac{\cos(n\theta)}{\cos\theta - \cos\theta_0} d\theta$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

can be written as

$$d_i(\theta) = \sum_1^N n \cdot A_n \cdot \frac{\sin(n\theta)}{\sin\theta}$$

We obtain  $C_{pi}$  as

$$C_{pi} = \frac{2b^2}{5} \int_0^\pi \sum_1^N A_n \cdot \sin(n\theta) \cdot \sum_1^N A_n \cdot n \cdot \sin(\theta) d\theta$$

From the standard integral.

$$\int_0^\pi \sin(n\theta) \cdot \sin(k\theta) d\theta = \int_0^\pi \sin(m\theta) \cdot \sin(k\theta) d\theta \begin{cases} 0 & m \neq k \\ \frac{\pi}{2} & m = k \end{cases}$$

$$C_{pi} = \frac{2b^2}{5} \cdot \sum_1^N (n A_n^2) \cdot \frac{\pi}{2} = 2 \cdot AR \cdot \sum_1^N (n A_n^2)$$

$$C_{pi} = \pi AR (A_1^2 + \sum_2^N n A_n^2) \\ = \pi AR \cdot A_1^2 \cdot \left[ 1 + \sum_2^N n \left(\frac{A_n}{A_1}\right)^2 \right]$$

$$C_{pi} = \frac{C_L^2}{\pi AR} (1 + \delta) \quad \text{where } \delta = \sum_2^N n \left(\frac{A_n}{A_1}\right)^2 \gg 0$$



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

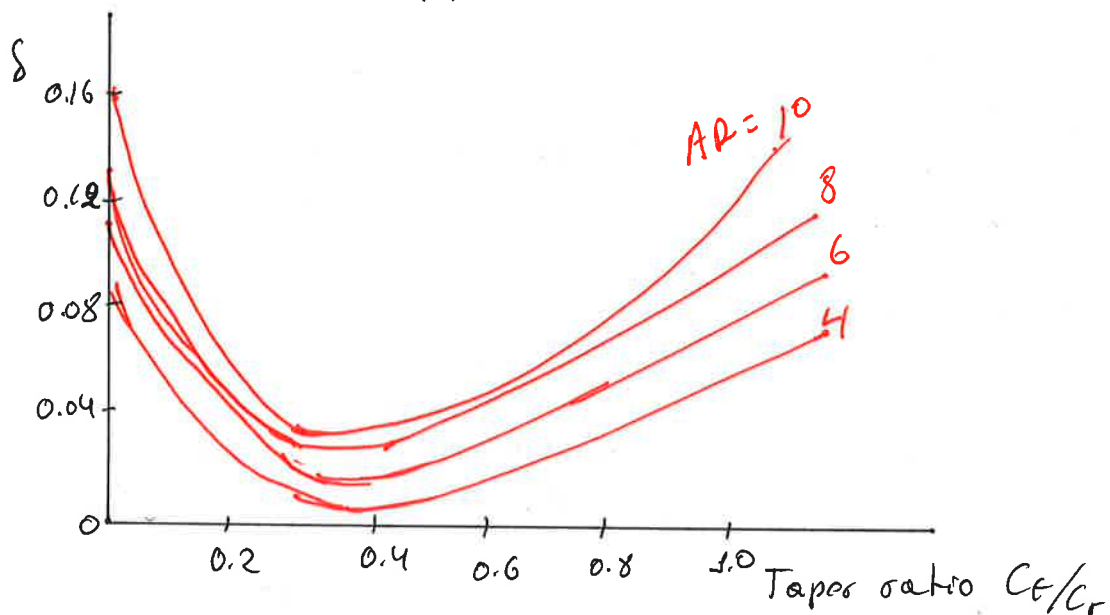
$$\delta = 0 \text{ and } e = 1$$

Hence the lift distribution which yields minimum induced drag is the elliptical lift distribution. This is why we have a practical interest in the elliptical lift distribution.

Elliptic wing: Elliptical planforms, are more expensive to manufacture.

Rectangulo. wing: Generates a lift distribution far from optimum.

Tapered wing: Can be designed with taper ratio, that is  $\text{Tip chord}/\text{root chord} \equiv C_e/C_r$ , such that the lift distribution closely approximates the elliptic case.



The variation of  $\delta$  as function of taper ratio for wings of different A.R. with this graph a tapered

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

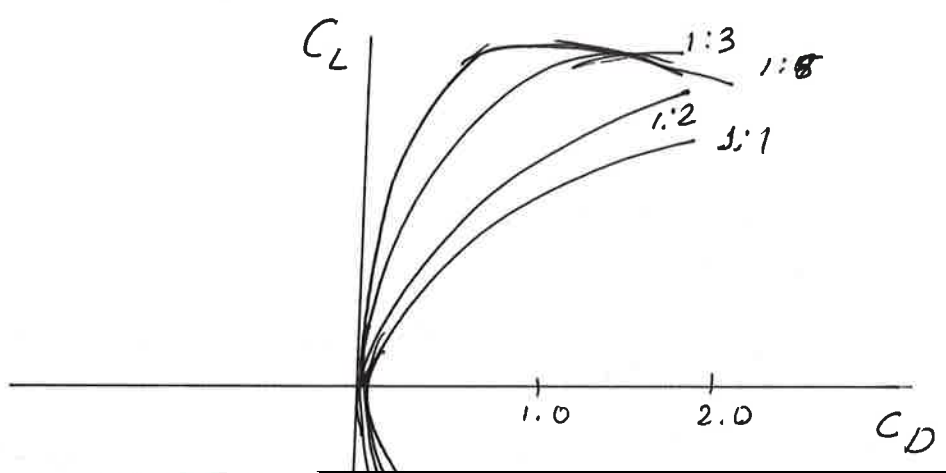


# Effect of Aspect Ratio

The  $AR$ . varies from 6 to 22 for standard subsonic airplanes and sailplanes. has much stronger effect on  $C_{Di}$  than the value of  $\delta$ . which from the above graph varies only by about 10 percent over the practical range of taper ratio. so that the primary design factor for minimizing induced drag is not the closeness to an elliptical lift distribution, but rather, the ability to make the aspect ratio as large as possible.

Recall that the total drag of finite wing is giving by

$$C_D = C_{d} + \frac{C_L^2}{\pi e AR}$$



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70  
 ---  
 ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

Consider two wings with different aspect ratio  $AR_1$  and  $AR_2$ . the Drag Coefficients  $C_{D1}$  and  $C_{D2}$  for the two wings as

$$C_{D1} = C_d + \frac{C_L^2}{\pi e AR_1}$$

$$C_{D2} = C_d + \frac{C_L^2}{\pi e AR_2}$$

Assume that the wings are at the same  $C_L$ .

Also since the airfoil section is the same for both wings.  $C_d$  is essentially the same.

Also the variation of  $e$  between the wing is only a few percent and can be ignored.

$$C_{D1} = C_{D2} + \frac{C_L^2}{\pi e} \left( \frac{1}{AR_1} - \frac{1}{AR_2} \right)$$

This equation can be used to scale the data of a wing with aspect ratio  $AR_2$  to correspond to the case of another aspect ratio  $AR_1$ .

Ex.

$$C_{D1} = C_{D2} + \frac{C_L^2}{\pi e} \left( \frac{1}{5} - \frac{1}{AR_2} \right)$$

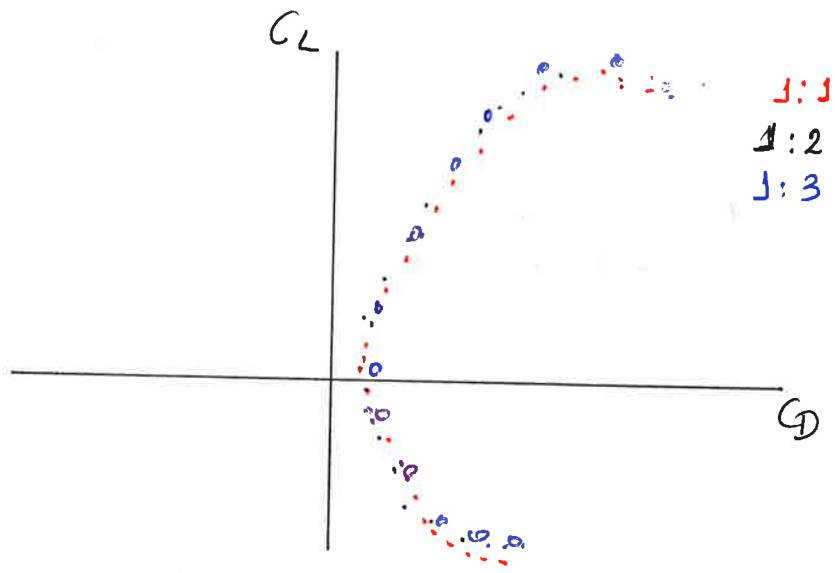
Inserting the value of  $C_{D2}$  and  $AR_2$  from the above graph. the resulting data for  $C_{D1}$  vs.  $C_L$  collapsed

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

Cartagena99



Airfoil and finite wing have some differences, finite wing produces a induced drag. Second major difference appears in the lift slope.

The slope of airfoil  $a_0 = \frac{dC_L}{d\alpha}$

The slope of wing  $a = \frac{dC_L}{d\alpha}$

Where  $a < a_0$ . The values of  $a_0$  and  $a$  are related as follows.

$$\frac{dC_L}{d(\alpha - \alpha_i)} = a_0$$

$$C_L = a_0 (\alpha - \alpha_i) + \text{constant}$$

$\alpha_i = \frac{C_L}{\pi AR}$  into



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70  
 ---  
 ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70

Differentiating.

$$-\frac{dC_L}{d\alpha} = a = \frac{a_0}{1 + a_0/\pi AR}$$

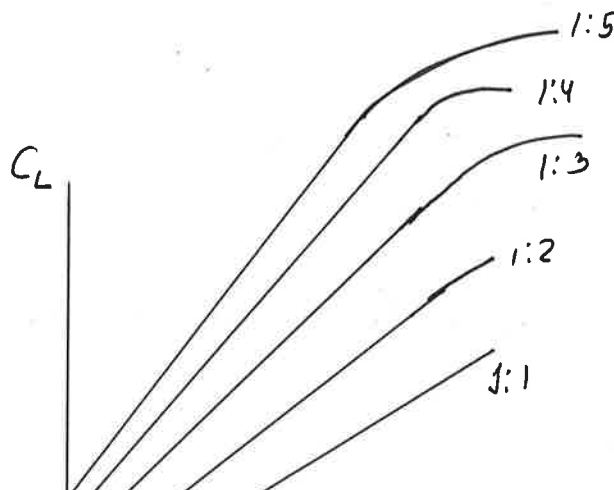
This equation gives the desired relation between  $a_0$  and  $a$  for an elliptic finite wing.

$$a = \frac{a_0}{1 + (a_0/\pi AR)(1 + \tau)}$$

This equation is for general planform, where  $\tau$  is a function of Fourier Coefficients  $A_n$ . The values of  $\tau$  range between 0.05 and 0.25.

$AR \rightarrow 0$ , a substantial difference can exist between  $a_0$  and  $a$ .

$AR \rightarrow \infty$   $a \rightarrow a_0$



Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

### Example

The analysis of this section is now applied to compute the characteristics of an untwisted rectangular wing  $AR = 6$  flying with  $\alpha$ .

Assume that the airfoil is uncambered so that the absolute of attack  $\alpha_a$  is equal to  $\alpha$  everywhere along the span.

$C = \epsilon_s$  and  $\alpha = \alpha_0 = 2\alpha$ , the equation of  $d$  is reduced to

$$d = \sum_{n=1}^{\infty} A_n \sin(n\theta) \left( 1 + \frac{n\pi}{2AR \sin\theta} \right)$$

For symmetrically loaded wing, the coefficients  $A_n$  vanish for even values of  $n$ .

$$d = A_1 \sin\theta \left( 1 + \frac{\pi}{6 \times 2 \sin\theta} \right) + A_3 \sin 3\theta \left( 1 + \frac{\pi}{4 \sin\theta} \right) + A_5 \sin 5\theta \left( 1 + \frac{5\pi}{12 \sin\theta} \right) + A_7 \sin 7\theta \left( 1 + \frac{7\pi}{12.5 \sin\theta} \right)$$

We take only one-half of the span because of

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70



$$\theta = \frac{\pi}{8}, \quad \theta = \frac{\pi}{4}; \quad \theta = \frac{3\pi}{8} \quad \text{and} \quad \theta = \frac{\pi}{2}$$

We obtain a set of equations.

$$0.644 A_1 + 2.8200 A_3 + 4.0841 A_5 + 2.2153 A_7 = d$$

$$0.9689 A_1 + 1.4925 A_3 + 2.0161 A_5 + 2.5397 A_7 = d$$

$$1.1857 A_1 + 0.7080 A_3 - 0.9249 A_5 + 2.7565 A_7 = d$$

$$1.261 A_1 - 1.7854 A_3 + 2.3090 A_5 - 2.8326 A_7 = d$$

$$A_1 = 0.9174 d \quad A_3 = 0.1104 d \quad A_5 = 0.0218 d \quad A_7 = 0.0038 d$$

The wing-lift coefficient is

$$C_L = \frac{\pi^2 A_1}{2} = 4.5273 d$$

Based on the values.

$$(C_{pi})_{el} = \frac{C_L^2}{\pi A R} = 1.0874 d^2$$

$$\text{and} \quad \delta = \frac{3 A_3^2 + 5 A_5^2 + 7 A_7^2}{A_1^2} = 0.0464$$

Induced Drag for the wing.

**Cartagena99**

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

# Calculating the lift coefficient

$$(C_L)_{el} = \frac{2\pi \cdot d}{1 + \frac{2\sigma}{\pi AR}} = 1.5\pi d = 4.7124d$$

So the slope of the lift coefficient curves,  $dC_L/d\alpha$  of rectangular wing is approximately 4% lower.

The result verifies the properties of arbitrary planforms are close to those of elliptical wing.

## Ex. 2

Consider a finite wing with an aspect ratio of 8 and taper ratio of 0.8. The airfoil section is thin and symmetric. Calculate the lift and induced drag coefficients for the wing when it is an angle of attack of  $5^\circ$ . Assume that  $\delta = \tau$

### Solution

from graph  $\delta = 0.055$  and  $\tau = \delta = 0.055$  and

$$a_0 = 2\pi$$

$$C_L = a_0 \alpha \rightarrow a = a_0$$



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
 LLAMA O ENVÍA WHATSAPP: 689 45 44 70  
 ---  
 ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
 CALL OR WHATSAPP: 689 45 44 70



Since airfoil is symmetric  $d_L = 0 = 0^\circ$

$$\text{Then } C_L = a \cdot d = (0.0867 \cdot \frac{5 \text{ Degree}}{\text{Degree}}) = 0.4335$$

$$C_{Di} = \frac{C_L^2}{\pi A R e} = 0.00789 \quad e = \frac{1}{1.5}$$

Ex. Jet transport patterned after cesna 560 citation V. shown a drag coefficient of cruise 0.015. In addition the zero-lift angle of attack is  $-2^\circ$ , the lift slope of the airfoil section is 0.1 per degree. The lift efficiency factor  $\Gamma = 0.04$  and the wing aspect ratio is 7.96. Calculate the angle of attack of the airplane.

Solution.

The lift slope of the airfoil is in radian

$$a_0 = 0.1 \text{ Per degree} = 0.1 \times (57.3) = 5.73 \text{ rad.}$$

$$a = \frac{a_0}{1 + (a_0 / \pi A R e) (1 + \Gamma)}$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

$$a = \frac{4.627}{57.3} = 0.0808 \text{ per degree}$$

$$C_L = a (\alpha - \alpha_{L=0})$$

Cruising Data. is :  $V = 790 \text{ km/h}$   $\rho = 0.41 \text{ kg/m}^3$   
 $W = 68 \text{ kN}$  Planform area =  $31.8 \text{ m}^2$

$$\text{Then } C_L = \frac{2W}{\rho V_\infty^2 S} = \frac{2 \cdot 68000}{(0.41) \cdot (219.4)^2 \cdot (31.8)} = 0.21$$

$$\alpha = \frac{C_L}{a} + \alpha_{L=0} = \frac{0.21}{0.0808} + (-2) = 0.6^\circ$$

Ex. Consider a rectangular wing on aspect ratio of 6. on induced drag factor  $\delta = 0.055$  and  $\alpha_{L=0} = -2^\circ$  and angle of attack of  $3.4^\circ$ .

The induced drag coefficient for this wing is 0.01. Calculate the induced drag coefficient for a similar wing. at the same angle of attack but with an AR. 10. Induced factors for drag and lift slope  $\delta$  and  $K$  are equal to each other AR 10,  $\delta = 0.105$

Solution



CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70  
...  
ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

For AR. 6 the  $C_L$  is

$$C_L^2 = \frac{\pi AR C_{Di}}{(1+f)} = \frac{\pi \cdot 6 \cdot (0.01)}{(1+0.055)} = 0.1787$$

$$C_L = 0.423$$

The lift slope of this wing is therefore

$$\frac{dC_L}{d\alpha} = \frac{0.423}{(3.4^\circ - (-2^\circ))} = 0.078/\text{degree} = 4.485/\text{rad}$$

The lift slope for the airfoil (the infinite wing).

$$\frac{dC_L}{d\alpha} = a = \frac{a_0}{1 + (a_0/\pi AR)(1+f)}$$

$$4.485 = \frac{a_0}{1 + \left(\frac{(1.055) \cdot a_0}{\pi \cdot 6}\right)} = \frac{a_0}{1 + 0.056a_0}$$

$$a_0 = \frac{5.989}{\text{rad}}$$

The second wing has the same airfoil section  
then  $a_0$  is the same.

The lift slope for the second wing.

$$a = \frac{a_0}{1 + ((1.055) \cdot a_0)} = 4.95/\text{rad}$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

The lift coefficient for the second wing is therefore (20)

$$C_L = a (\alpha - \alpha_{L=0}) = 0.086 [3.4^\circ - (-2^\circ)]$$

$$C_L = 0.464$$

In turn, the induced drag coefficient is

$$C_{Di} = \frac{C_L^2}{\pi AR} (1+\delta) = \frac{0.464^2 (1+0.105)}{\pi \cdot 10} = 0.0076$$

Consider same  $C_L$  rather than  $\alpha$  for both wings.

$$C_{Di} = \frac{C_L^2}{\pi AR} (1+\delta) = \frac{(0.423)^2 (1+0.105)}{\pi \cdot 10} = 0.0063.$$

One observation about induced Drag  $D_i$ ; in contrast to the induced drag coefficient  $C_{Di}$ , that  $C_{Di}$  decreases by increases the AR. The Drag ( $D_i$ ) is governed by other parameter.

$$D_i = \frac{q}{\rho} S \cdot C_{Di} = \frac{q}{\rho} S \cdot \frac{C_L^2}{\pi AR} \quad \text{for steady flight}$$

$$C_L^2 = \left( \frac{L}{S} \right)^2 - \left( \frac{W}{S} \right)^2$$

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70

$D_i = \frac{L}{\pi e} \frac{1}{q_{\infty}} \left( \frac{W}{b^2} \right)$  For steady flight induced  
Drag does not depend on the

aspect ratio; but rather other parameter  $\left( \frac{W}{b} \right)$   
called the <sup>span</sup> loading: span loading =  $\frac{W}{b}$ . Drag (induced)  
can be reduced by increasing b. span.

How much of the total drag of an airplane  
is induced drag?

The parasite drag for a generic subsonic  
jet transport is the sum of the drag due to  
skin friction and pressure drag due to flow  
separation associated with the complete airplane,  
including wing. At cruise about 25% is induced  
drag but at takeoff can be 60% or more  
of the total drag.

Cartagena99

CLASES PARTICULARES, TUTORÍAS TÉCNICAS ONLINE  
LLAMA O ENVÍA WHATSAPP: 689 45 44 70

---

ONLINE PRIVATE LESSONS FOR SCIENCE STUDENTS  
CALL OR WHATSAPP: 689 45 44 70